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Modification on Restrictive Taylor and Padé Approximations

M. S. Abdelwahed [a], Tamer M. Rageh [a*], Aisha F. Fareed [a] and Ahmed M. Youssef [a]

[a] *Department of Basic Science Engineering, Faculty of Engineering in Benha, Benha University, Benha 13512, Egypt.*

Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

We review Taylor approximation (TA), Padé approximation (PA), Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA). After comparing these four approximation methods with two other modified approximation methods: Modified Restrictive Taylor approximation (MRTA) and Modified Restrictive Padé approximation (MRPA), we give test examples to illustrate how the modified approximations could be used. The mathematical principles behind all these approximations could be applied for the development of new computing methods.

Keywords: *Taylor polynomial; padé approximation; restrictive Taylor approximation; restrictive Padé approximation; modified restrictive Taylor approximation and modified restrictive Padé approximation.*

1 Introduction

"The relationship between the coefficients of a series expansion and of the value of the function is both important mathematically and significant practically. It is fundamental to the study of mathematical analysis as well as to the actual calculation of mathematical models of nature in many applied sciences and engineering challenges, such as the transfer function of electrical and dynamic systems, biology, physics, chemistry, and

**Corresponding author: Email: Tamer_m_rageh@yahoo.com, tamer.rageh@bhit.bu.edu.eg;*

medicine. "Padé approximant [1-3] is typical rational function of a given order. Under this technique, the approximant's power series agrees with the power series of the function it is approximating". Henri Padé developed the method about 1890, although it was Georg Frobenius who initially proposed it and looked into the characteristics of rational power series approximations. This approach outperforms rational function [4]. Brezinski [5] introduced the Padé approximant, Khan [6] used the Padé -Hermite approximant, and Ysern and Lagomasino [7] demonstrated the convergence of multipoint Padé -type approximants.

Ismail and Elbarbary introduced the restrictive Taylor approximation to solve Parabolic Partial Differential Equations [8], Ismail et al. [9,10] approximated "a solution for the Convection-Diffusion equation and the KdV-Burgers equation", and Rageh et al. [11] used "RTA to solve Gardner and KdV equations".

Restrictive Padé approximation was used by Ismail et al. [12-15] to solve the Schrodinger problem and Generalized Burger's equation. Ismail [16] examines how (RPA) converges to the precise IBVP solutions of the parabolic and hyperbolic kinds. Solvability and uniqueness were introduced by Ismail [17] for both (RTA) and (RPA).

2 Classical and Restrictive Approximations

2.1 Taylor and Padé approximations

2.1.1 Taylor series

Taylor series is a polynomial of infinite degree that can be used to represent many different functions, particular functions that are not polynomial. This means that Taylor expansion is a method through which we can convert a non-polynomial function into a polynomial function. Taylor approximation is useful and preferably to be used, since polynomials are easier to be evaluated at some particular values. It is also easier to differentiate and integrate. Then the Taylor series of a real function $f(\xi)$ about a point a is given by

$$f(\xi) = f(a) + \frac{(\xi - a)}{1!} f'(a) + \frac{(\xi - a)^2}{2!} f''(a) + \cdots \tag{1}$$

where;

$a \in R$ and $f(\xi)$ is infinity differentiable function on the interval I that contains the point a .

For evaluating finite number of terms, the Taylor polynomial of degree n is defined by

$$f(\xi) = \sum_{k=0}^{n} \frac{(\xi - a)^k f^{(k)}(a)}{k!} + R_n \tag{2}$$

where R_n is the Lagrange remainder term given by

$$R_n(\xi) = \frac{(\xi - a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi), \qquad a \leq \varepsilon \leq \xi \tag{3}$$

So that the maximum error after n terms of Taylor expansion is the maximum value obtained from equation (3) running through all $\varepsilon \in [a, \xi]$.

2.1.2 Padé approximation

Padé Approximation approximates a function using the rational polynomials. Since the Taylor series use repeated differentiation to produce a polynomial approximation of a function then, Taylor series often cannot be used to extrapolate the function for very long before becoming rapidly diverging. Padé approximants often follow the function more closely for longer. This technique was developed around 1890 by Henri Padé (1863-1953).

For a given function $f(\xi)$ and two integers $m \geq 0$ and $n \geq 1$ the Padé approximant (PA) of order $[m/n]$ is

$$f_{[m/n]}(\xi) = \frac{a_{0+}a_1\xi + a_2\xi^2 + \cdots + a_M\xi^m}{b_{0+}b_1\xi + b_2\xi^2 + \cdots + b_N\xi^n} \tag{4}$$

Since PA are rational functions, with a denominator that does not vanish at zero, and whose series expansion matches a given series as far as possible, then its coefficients are determined from the condition

$$(b_{0+}b_1\xi + b_2\xi^2 + \cdots + b_n\xi^n)(c_0 + c_1\xi + c_2\xi^2 + \cdots) = a_0 + a_1\xi + a_2\xi^2 + \cdots + a_m\xi^m + O(\xi^{m+n+1})$$

with coefficient condition $b_0 = 1$, then

$$a_0 = c_0$$
$$a_1 = c_1 + b_1 c_0$$
$$a_m = c_m + \sum_{k=1}^{p} b_i\, c_{m-k}$$

where $p = \min(m, n)$, and the coefficients c_i are obtained from the expansion of the function $f(\xi)$ in Maclaurin series.

2.2 Restrictive type approximations

In the following pages, we describe two forms for "*Restrictive*" type approximations in which we find the restrictive parameter that gives the exact solution at a given number of points in some type of problems.

2.2.1 Restrictive Taylor approximation (RTA)

Consider a function $f(x)$ defined in a neighborhood of $x = a$ and it has derivatives up to order $(n + 1)$ Constructing a function

$$RT_{n,f(x)}(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{\varepsilon(x-a)^n}{n!}f^{(n)}(a) \tag{5}$$

where ε is a parameter to be determined by adding the following condition [9-11]:

$$RT_{n,f}(x_a) = f(x_a)$$

Let x_a be some points in the domain of the function f. The function $RT_{n,f}(x)$ is called restrictive Taylor approximation of order n of the function $f(x)$ at the point $x = a$. The following theorem gives the value of the remainder term of this approximation.

Assume the function $f(x)$ and its derivatives up to an order $n + 1$ are continuous in a certain neighborhood of a point. Suppose, furthermore, that x is any value of the argument from the indicated neighborhood and ε is the restrictive Taylor parameter, then there is a point ξ which lies between the points a and x such that the formula:

$$f(x) = RT_{n,f}(x) + \square_{n+1}(x, \varepsilon) \tag{6}$$

is true, for which

$$\square_{n+1}(x, \varepsilon) = \frac{\varepsilon(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi) - \frac{n(\varepsilon-1)^{n+1}(x-a)^{n+1}}{(n+1)!\,(x-\xi)}f^{(n)}(\xi) \tag{7}$$

$, \xi \in (a, x)$

where $\square_{n+1}(x, 1)$ is the Taylor remainder term.

2.2.2 Restrictive Padé approximation (RPA)

We construct a restrictive type of Padé approximation [13,14] of the function $f(x)$ with a parameter yet to be determined. If it reduces the remainder term to zero, we will get the classical Padé approximation. The restrictive Padé approximation is a rational function in the form [12]:

$$RPA[m+\alpha/n]_f\,(x) = \frac{\sum_{i=0}^{m} a_i x^i + \sum_{i=1}^{\alpha} \varepsilon_i x^{m+i}}{1 + \sum_{i=1}^{n} b_i x^i} \tag{8}$$

where the positive integer α does not exceed the degree of the numerator,

$$\alpha = 0(1)n$$

Such that

$$f(x) = RPA[m+\alpha/n]_f\,(x) + O(X^{m+n+1}) \tag{9}$$

and let f(x) has a Maclaurin series expansion

Table 1. Restrictive Padé table

$[\alpha/0]$		$[\alpha$	$[\alpha$	...		$[\alpha$	...
	$[\alpha$	$[\alpha$	$[\alpha$	...		$[\alpha$	...
	$[\alpha$	$[\alpha$	$[\alpha$	...		$[\alpha$	...
$\vdots$		$\vdots$	$\vdots$	$\ddots$	$\vdots$		...
$[\alpha+m/0]$	$[\alpha+m/1]$	$[\alpha+m/2]$]		...	$[\alpha+m/n]$]		...
$\vdots$		$\vdots$	$\vdots$		$\vdots$		

Table 1 is the Restrictive Padé table for $f(x)$. The first j[th] columns disappear when $\alpha > j-1$. The case $\alpha = 0$ gives the classical Padé table.

The case $\alpha = 1$ gives the following selected elements of RPA table:

$$RPA[1/1]_f\,(x) = \frac{a_0 + a_1 x}{1 + b_1 x}$$

where

$$a_0 = c_0$$
$$a_1 = \varepsilon_1$$
$$b_1 = \frac{\varepsilon_1 - c_1}{c_0}$$
$$RPA[2/1]_f\,(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x}$$

where

$$a_0 = c_0$$
$$a_1 = c_1 + \frac{c_0 + (\varepsilon_1 - c_2)}{c_1}$$
$$a_2 = \varepsilon_1$$

$$b_1 = \frac{\varepsilon_1 - c_2}{c_1},$$
$$RPA[3/1]_f(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x}$$

where

$$a_0 = c_0$$
$$a_1 = c_1 + \frac{c_0 + (\varepsilon_1 - c_2)}{c_1}$$
$$a_2 = \varepsilon_2 + \frac{c_1(\varepsilon_1 - c_3)}{c_2}$$
$$a_3 = \varepsilon_1$$
$$b_1 = \frac{\varepsilon_1 - c_3}{c_2}$$

3 Concave and Convex functions

We use concave and convex to describe the shape or the curvature of the curve. The definitions concavity and convexity of a single variable function are widely used in economic theory and are also central to optimization theory. The function $f(x)$ is said to be a concave function if every line segment joining any two points on the curve of this function lie below the graph at any point. On the other hand, a function $f(x)$ is said to be convex if every line segment joining two points on the curve of the function lie above the graph at any point. See Fig. 1 below:

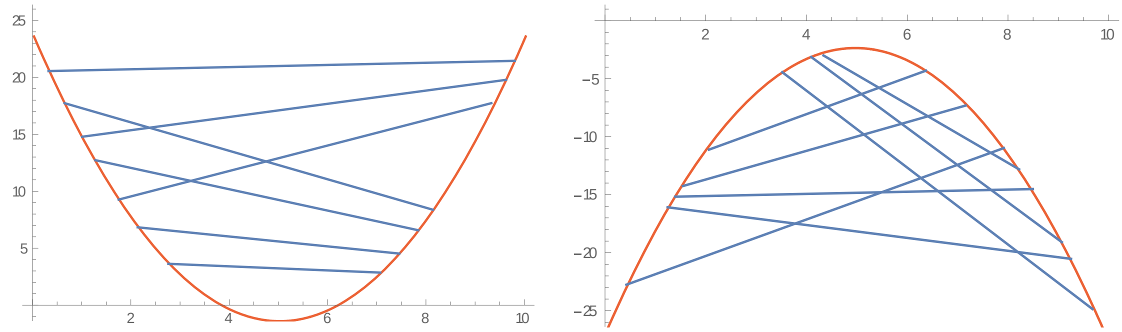


Fig. 1. Convex (on left) and concave (on right) functions

We can easily conclude that the graph of any differentiable concave function lies below the tangent at pointx_0, which its slope is $f'(x_0)$. Similarly, the graph of any differentiable convex function lies above the tangent at point x_0, which its slope is $f'(x_0)$. Thus, the definition of the concave and convex function may be written as follows:

Definition:

Let f is a single variable function defined on the interval I**.** Then the function f is

Concave function if for all $x \in I, all\ y \in I$ and $\lambda \in [0,1]$ we have

$$f((1+\lambda)x + \lambda y) \geq (1+\lambda)f(x) + \lambda f(y)$$

Convex function if for all $x \in I, all\ y \in I$ and $\lambda \in [0,1]$ we have

$$f((1+\lambda)x + \lambda y) \leq (1+\lambda)f(x) + \lambda f(y)$$

Both convex and concave function if for all $x \in I, all\ y \in I$ and $\lambda \in [0,1]$ we have

$$f((1+\lambda)x+\lambda y)=(1+\lambda)f(x)+\lambda f(y)$$

Thus, the function is both convex and concave if and only if it is linear function.

In this paper our approximated functions (RTA& RPA) will be classified into either concave or convex function. We are looking forward to reducing the maximum norm error L_∞ resulting from our approximated functions (RTA& RPA) by using this classification as we will show in the next part.

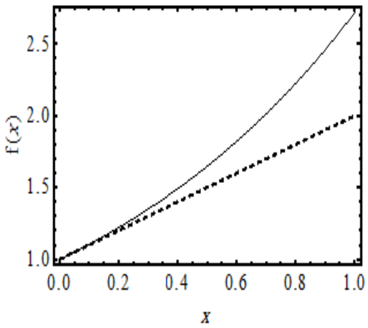
4 Modification in restrictive Taylor and Padé approximation

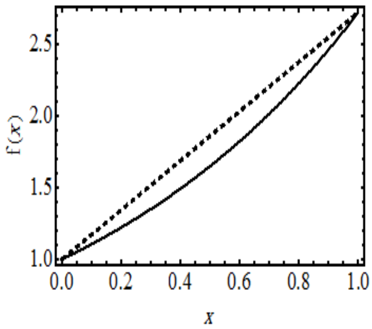
After we use the restrictive approximation we find the more it converges, the larger the maximum L_∞ error norm we obtain. The new modification is to eliminate this max error and to force the absolute error to be near zero around the region that has the maximum error.

From the analysis of error and plotting of the absolute error between the function needed to be approximated and the restrictive approximation, we can classify the modification into two types according to whether the function is convex or concave.

4.1 Case 1: Convex functions

From Fig. 2b, the RA lies above the function so the error will be negative; we add the modification term.





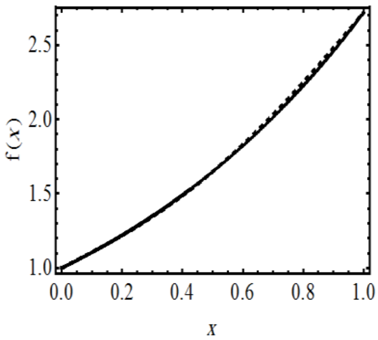
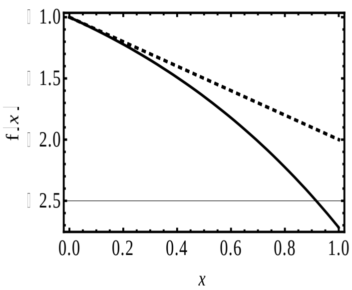


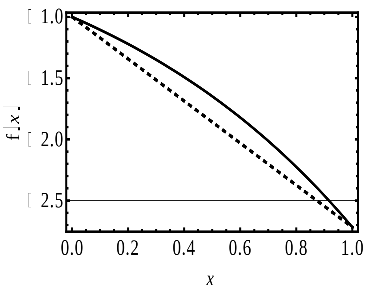
Fig. 2a **Fig. 2b** **Fig. 2c**

Fig. 2(a-c). Convex functions

4.2 Case 2: Concave functions

From Fig. 3b the RA lies below the function so the error will be positive; we subtract the modification term.





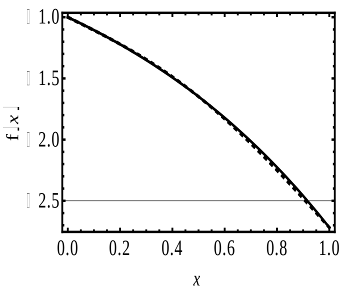


Fig. 3a **Fig. 3b** **Fig. 3c**

Figs. 3(a-c). Concave functions

4.2.1 We choose the modification term to satisfy three conditions

1) At $x = 0$, the modification term vanish (equal zero). See point A in Fig. 4.
2) At the point B, the maximum error (L_∞ error norm) the modification term has maximum value and equal to the L_∞ error norm; the new approximations is used to force the error to be near zero around the point having the L_∞ error norm.
3) At x_a where we calculate the restrictive term the error equal to zero, so we do not need the modification term at point C in Fig. 4.

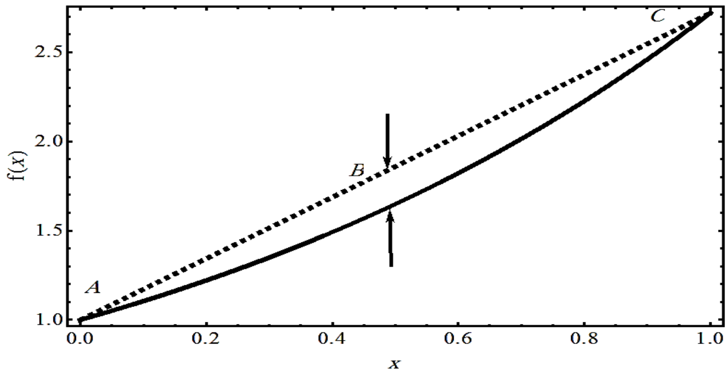


Fig. 4. Error pints

5 Numerical examples

In this section we introduce two examples solved by Taylor approximation (TA), Padé approximation (PA), Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA). Then comparing this result by the two Modified restrictive Taylor approximation (MRTA) and Restrictive Padé approximation (MRPA). Figs. 5 – 6 show the function used together with the expansions and errors.

Example 1.

$$f(x) = exp(cx) \tag{10}$$

For $c = 1$ the restrictive terms are calculated at $x = 1$ for both RTA and RPA. We limit the expansion domain to [0, 1] and plot TA, PA, RTA, RPA, MRTA and MRPA in Fig. 5. Also, the errors are in Fig. 6.

$$TA_1 = 1 + x$$
$$RTA_1 = 1 + \varepsilon x$$

where ε is the restrictive term to be determined by

$$RTA_1 = 1 + 1.718281828459045x$$
$$MRTA_1 = RTA_1 - L_\infty \operatorname{Sin}(x)$$
$$MRTA_1 = 1 + 1.718281828459045x - 0.21041964352939435 \operatorname{Sin}(x)$$

Table 2. Error norm for Taylor expansions for example 1

	TA_1	**RTA_1**	**$MRTA_1$**
L_2	1.0915436502789486	0.488301679070951	0.039990197968343945
L_∞	0.7182818284590451	0.21041964352939435	0.02540297101555078

$$PA_1 = \frac{1+\frac{x}{2}}{1-\frac{x}{2}} \qquad RPA_1 = \frac{1+\varepsilon x}{1-b_1 x}$$

where ε is the restrictive term to be determined by

$$RPA_1 = \frac{1 + 0.5819767068693265x}{1 - 0.41802329313067355x}$$

$$MRPA_1 = RPA_1 + L_\infty \operatorname{Sin}(x)$$

$$MRPA_1 = \frac{1 + 0.5819767068693265x}{1 - 0.41802329313067355x} + 0.02419074111857622 \operatorname{Sin}(x)$$

Table 3. Error norm for Padé expansions for example 1

	PA_1	RPA_1	$MRPA_1$
L_2	0.35755397907149267	0.048375725339978484	0.028051263981545418
L_∞	0.2817181715409549	0.02419074111857622	0.01522650542301296

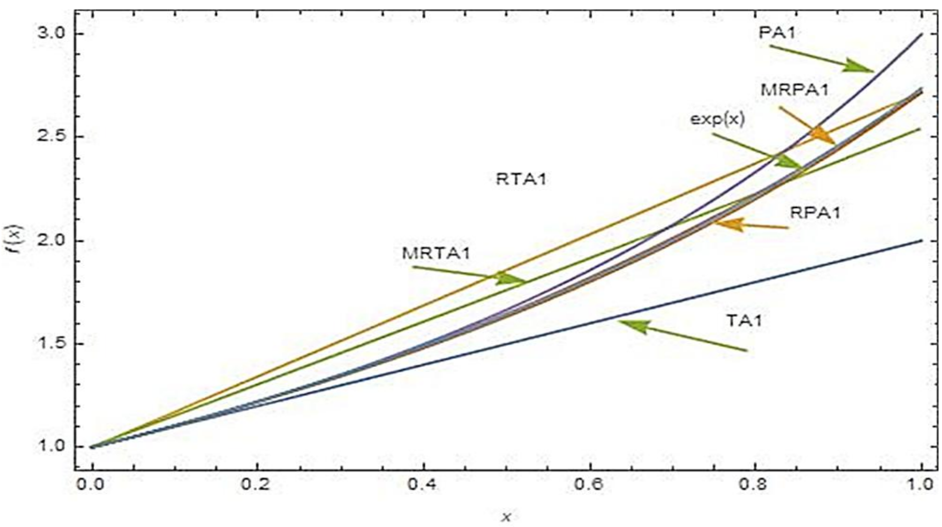


Fig. 5. Expansion of example 1

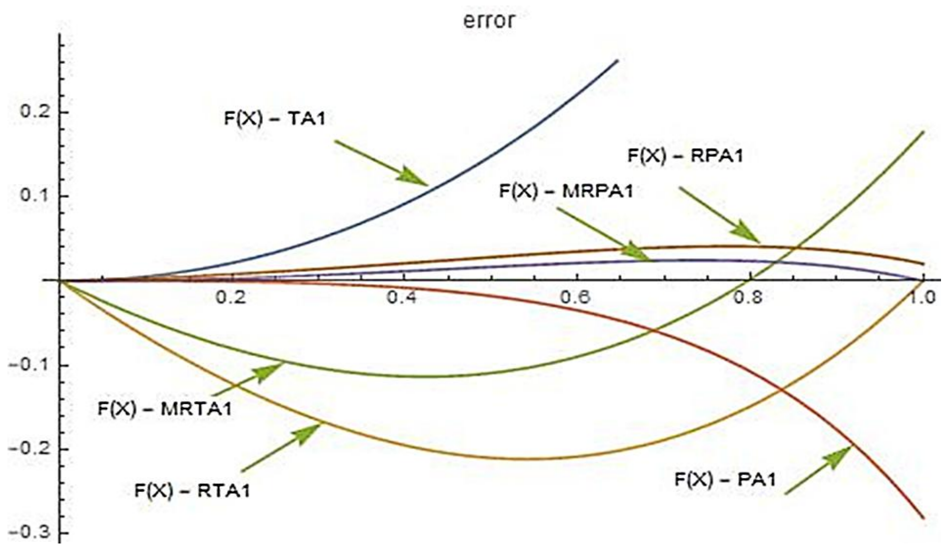


Fig. 6. The errors in example 1

Table 4. Error in example 1

x	TA_1	PA_1	RTA_1	RPA_1	$MRTA_1$	$MRPA_1$
0	0	0	0	0	0	0
0.1	0.005170918075647624	−0.0000922398190890128	−0.0666572647702568	0.0008083183123754889	−0.0016340189693555551	−0.006667031799789447
0.2	0.0214027581601699	−0.0008194640620524662	−0.12225360753163916	0.003156340953204495	0.0014279557275782562	−0.011062619918319694
0.5	0.1487212707001282	−0.017945395966538324	−0.21041964352939435	0.016600711871570306	0.	−0.007590029247005914
0.7	0.31375270747047646	−0.0631703694526009	−0.18904457245085515	0.02419074111857622	−0.018811504885256625	0.004620020447123174
0.9	0.5596031111569499	−0.1767605252066864	−0.08685053445619051	0.01678450425391409	−0.021827288655289045	0.009309154141749154
1	0.7182818284590451	−0.2817181715409549	0	0	0	0

Table 5. Error norm for Taylor expansions for example 2

	TA_1	RTA_1	$MRTA_1$
L_2	0.8045311072982965	0.1522780428123673	0.02842734952458836
L_∞	0.4571067811865476	0.06634613154689295	0.016018818139471036

Table 6. Error Norm for Padé expansions for example 2

	PA_1	RPA_1	$MRPA_1$
L_2	0.012674300329633452	0.002665666175542447	0.0001662632781343035
L_∞	0.007178933099166618	0.001204511854803969	0.00008268011675605091

Table 7. Error in example 2

x	TA_1	PA_2	RTA_1	RPA_1	$MRTA_1$	$MRPA_1$
0	0	0	0	0	0	0
0.1	0.010414346693485399	−0.000069524274256327 35	−0.0352963314251693	0.00028953451630442384	−0.014794249266143589	−0.0000826801167560509 1
0.2	0.03640526042791836	−0.00038719240227047536	−0.05501609580939115	0.000723024244747017	−0.016018818139471036	0.000015029940281796605
0.5	0.16556941504209488	−0.002534033233767219	−0.06298397555117885	0.001204511854803969	0.0033621559957141	0
0.7	0.2750000000	−0.004385964912280715	−0.04497474683058322	0.0009005993084881814	0.008700401101888966	−0.00007387125197433342
0.9	0.3946229171289245	−0.006265407744171947	−0.01677318593896826	0.0003355682464551979	0.003728896220057454	−0.00003664638660527686
1	0.4571067811865476	−0.007178933099166618	$1.110223024625156 \times 10^{-16}$	0	$1.110223024625156 \times 10^{-16}$	0

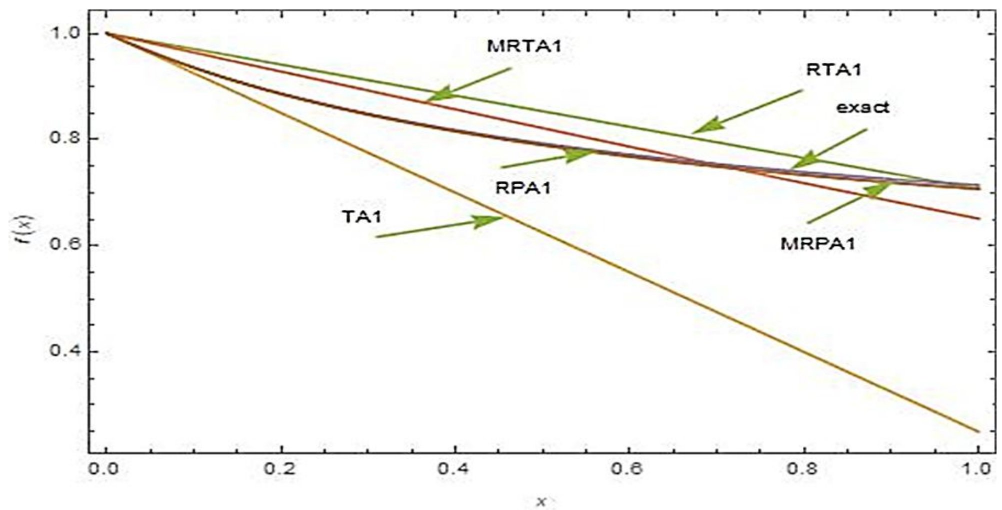


Fig. 7. Expansion of example 2

Example 2.

$$f(x) = \sqrt{\frac{1 + \frac{1}{2}x}{1 + 2x}} \tag{11}$$

Baker and Morris [1] have shown the accurate of PA via TA in this example. We introduce a better method (RTA and RPA) with results as shown in Fig. 7 and the error in Fig. 8.

$$TA_1 = 1 - 0.75\,x \qquad RTA_1 = 1 + \varepsilon x$$

where ε is the restrictive term to be determined by

$$RTA_1 = 1 - 0.2928932188x \qquad MRTA_1 = RTA_1 - L_\infty \operatorname{Sin}(x)$$
$$MRTA_1 = 1 - 0.2928932188x - 0.06634613154689295 \operatorname{Sin}(x)$$

$$PA_1 = \frac{1+0.875x}{1+1.625x} \qquad RPA_1 = \frac{1+\varepsilon x}{1-b_1 x}$$

where ε is the restrictive term to be determined by

$$RPA_1 = \frac{1 + 0.8106601718x}{1 + 1.560660172x}$$

$$MRPA_1 = RPA_1 + L_\infty \operatorname{Sin}(x)$$

$$MRPA_1 = \frac{1 + 0.8106601718x}{1 + 1.560660172x} + 0.001204511854803969 \operatorname{Sin}(x)$$

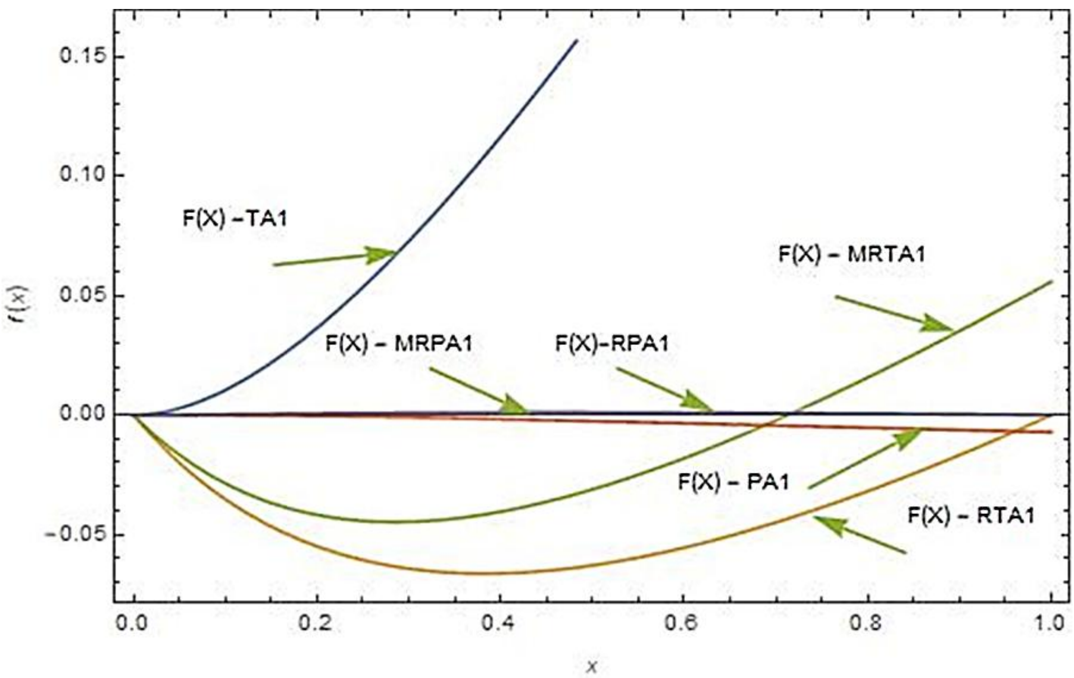


Fig. 8. Errors in example 2

6 Conclusion

The numerical examples show better accurate for Modified Restrictive Taylor approximation (MRTA) and Modified Restrictive Padé approximation (MRPA) than Restrictive Taylor approximation (RTA) and Restrictive Padé approximation (RPA). We know that TA and PA gives the exact solution at also at the point used to calculate the restrictive term as shown in Fig. 2.b or Fig. 3.b. As we show in example 1 and 2 (Figs. 5, 6, 7 and 8) we use RTA and RPA to force the approximation to be close to the exact curve by adding modified terms so that the error is close to zero. The mathematical principles behind all these approximations and their modifications could be applied for the development of new computing methods.

Competing Interests

Authors have declared that no competing interests exist.

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